AN INTEGRAL FORM OF THE NONLINEAR SCHRÖDINGER EQUATION WITH VARIABLE COEFFICIENTS

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ABSTRACT. We discuss an integral form of the Cauchy initial value problem for the nonlinear Schrödinger equation with variable coefficients. Some special and limiting cases are outlined.

1. Introduction

In the previous Letter we have constructed the time evolution operator for the linear one-dimensional time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=H\left(t\right)\psi \tag{1.1}$$

in a general case when the Hamiltonian is an arbitrary quadratic form of the operator of coordinate and the operator of linear momentum; see [15]. In this approach, several exactly solvable models, that have been studied elsewhere, are classified in terms of elementary solutions of certain characteristic equation. A particular solution of the corresponding nonlinear Schrödinger equation with variable coefficients had been obtained in a similar fashion. In this paper we rewrite a nonlinear Schrödinger equation

$$\left(i\frac{\partial}{\partial t}-H\left(t\right)\right)\psi\left(x,t\right)=F\left(t,x,\psi\left(x,t\right)\right) \tag{1.2}$$

in an integral form and consider several examples. In general, we do not assume that the time-dependent linear Hamiltonian $H\left(t\right)$ here is the Hermitian operator.

2. A General Lemma: Duhamel's Principle

The following result is helpful in study of solutions of the nonlinear Schrödinger equation (1.2) by a fixed point argument.

Lemma 1. *Suppose that the Cauchy initial value problem*

$$\left(i\frac{\partial}{\partial t}-H\left(t\right)\right)\psi_0\left(x,t\right)=0,\qquad \psi_0\left(x,t\right)\big|_{t=0}=\varphi\left(x\right) \tag{2.1}$$

for a linear time-dependent Schrödinger equation can be solved in terms of the time evolution operator

$$\psi_0\left(x,t\right)=U\left(t\right)\psi_0\left(x,0\right) \tag{2.2}$$

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with

$$\left(i\frac{\partial}{\partial t}-H\left(t\right)\right)U\left(t\right)=0 \tag{2.3}$$

and

$$U\left(0\right)=I=U\left(t\right)U\left(t\right)^{-1}. \tag{2.4}$$

Then the initial value problem

$$\left(i\frac{\partial}{\partial t}-H\left(t\right)\right)\psi=F\left(t,x,\psi\right),\qquad \left.\psi\left(x,t\right)\right|_{t=0}=\chi\left(x\right) \tag{2.5}$$

for the nonlinear Schrödinger equation can be rewritten as an integral equation

$$\psi\left(x,t\right)=U\left(t\right)\psi\left(x,0\right)-i\int_0^t U\left(t\right)U\left(s\right)^{-1}F\left(s,x,\psi\left(x,s\right)\right)\ ds \tag{2.6}$$

in terms of the time evolution operator $U\left(t\right)$ for the linear equation and its inverse.

Proof. Indeed, from (2.6)

$$\begin{aligned}\left(i\frac{\partial}{\partial t}-H\left(t\right)\right)\psi&=\left(i\frac{\partial}{\partial t}-H\left(t\right)\right)U\left(t\right)\psi\left(x,0\right)\\&+\frac{\partial}{\partial t}\int_0^t U\left(t\right)U\left(s\right)^{-1}F\left(s,x,\psi\left(x,s\right)\right)\ ds\\&+i\int_0^t H\left(t\right)U\left(t\right)U\left(s\right)^{-1}F\left(s,x,\psi\left(x,s\right)\right)\ ds,\end{aligned}$$

where

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^t U\left(t\right)U\left(s\right)^{-1}F\left(s,x,\psi\left(x,s\right)\right)\ ds\\&=U\left(t\right)U\left(t\right)^{-1}F\left(t,x,\psi\left(x,t\right)\right)\\&\quad+\int_0^t\frac{\partial U\left(t\right)}{\partial t}U\left(s\right)^{-1}F\left(s,x,\psi\left(x,s\right)\right)\ ds.\end{aligned}$$

Thus, by (2.3) and (2.4)

$$\begin{aligned}\left(i\frac{\partial}{\partial t}-H\left(t\right)\right)\psi&=F\left(t,x,\psi\left(x,t\right)\right)\\&-i\int_0^t\left(i\frac{\partial}{\partial t}-H\left(t\right)\right)U\left(t\right)\left(U\left(s\right)^{-1}F\left(s,x,\psi\left(x,s\right)\right)\right)\ ds\\&=F\left(t,x,\psi\left(x,t\right)\right).\end{aligned}$$

Initial conditions are satisfied in view of

$$\begin{aligned}&\lim_{t\to0^+}\int_0^t U\left(t\right)U\left(s\right)^{-1}F\left(s,x,\psi\left(x,s\right)\right)\ ds\\&=U\left(0\right)\lim_{t\to0^+}\int_0^t U\left(s\right)^{-1}F\left(s,x,\psi\left(x,s\right)\right)\ ds=0,\end{aligned}$$

where we have used the fact that if $\lim_{t\to0^+} f(t) = f(0)$, then

$$\lim_{t\to0^+} \int_0^t f(s)\ ds = 0.$$

This completes the proof. $\square$

When F does not depend on ψ, expression (2.6) solves the Cauchy initial value problem (2.5) for the corresponding nonhomogeneous Schrödinger equation.

The integral equation (2.6) can also be rewritten in the form

$$\begin{aligned} \psi(x,0) &= U^{-1}(t)\,\psi(x,t) + i\int_0^t U(s)^{-1} F(s,x,\psi(x,s))\ ds \\ &= T(t)^{-1}\psi(x,t), \end{aligned} \tag{2.7}$$

which gives explicitly the inverse of the time evolution operator

$$\psi(x,t) = T(t)\,\psi(x,0) \tag{2.8}$$

for the nonlinear Schrödinger equation (1.2). Thus, in general, solution of the Cauchy initial value problem (2.5) is a problem of inversion of this nonlinear integral operator.

3. Quadratic Hamiltonians

The fundamental solution of the linear Schrödinger equation with the quadratic Hamiltonian of the form

$$i\frac{\partial\psi}{\partial t} = -a(t)\frac{\partial^2\psi}{\partial x^2} + b(t)\,x^2\psi - i\left(c(t)\,x\frac{\partial\psi}{\partial x} + d(t)\,\psi\right) - f(t)\,x\psi + ig(t)\frac{\partial\psi}{\partial x}, \tag{3.1}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, and $g(t)$ are given real-valued functions of time t only, can be found with the help of a familiar substitution

$$\psi = Ae^{iS} = A(t)\,e^{iS(x,y,t)}, \qquad A = A(t) = \frac{1}{\sqrt{2\pi i\mu(t)}} \tag{3.2}$$

with

$$S = S(x,y,t) = \alpha(t)\,x^2 + \beta(t)\,xy + \gamma(t)\,y^2 + \delta(t)\,x + \varepsilon(t)\,y + \kappa(t), \tag{3.3}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$, $\varepsilon(t)$, and $\kappa(t)$ are differentiable real-valued functions of time t only; see [15]. Indeed,

$$\frac{\partial S}{\partial t} = -a\left(\frac{\partial S}{\partial x}\right)^2 - bx^2 + fx + (g - cx)\frac{\partial S}{\partial x} \tag{3.4}$$

provided

$$\frac{\mu'}{2\mu} = a\frac{\partial^2 S}{\partial x^2} + d = 2\alpha(t)\,a(t) + d(t). \tag{3.5}$$

Equating the coefficients of all admissible powers of $x^m y^n$ with $0 \le m+n \le 2$, gives the following system of ordinary differential equations

$$\frac{d\alpha}{dt} + b(t) + 2c(t)\,\alpha + 4a(t)\,\alpha^2 = 0, \tag{3.6}$$

$$\frac{d\beta}{dt} + (c(t) + 4a(t)\,\alpha(t))\,\beta = 0, \tag{3.7}$$

$$\frac{d\gamma}{dt}+a(t)\beta^2(t)=0, \tag{3.8}$$

$$\frac{d\delta}{dt}+(c(t)+4a(t)\alpha(t))\delta=f(t)+2\alpha(t)g(t), \tag{3.9}$$

$$\frac{d\varepsilon}{dt}=(g(t)-2a(t)\delta(t))\beta(t), \tag{3.10}$$

$$\frac{d\kappa}{dt}=g(t)\delta(t)-a(t)\delta^2(t), \tag{3.11}$$

where the first equation is the familiar Riccati nonlinear differential equation; see, for example, [23], [38], [45] and references therein. Substitution of (3.5) into (3.6) results in the second order linear equation

$$\mu''-\tau(t)\mu'+4\sigma(t)\mu=0 \tag{3.12}$$

with

$$\tau(t)=\frac{a'}{a}-2c+4d, \qquad \sigma(t)=ab-cd+d^2+\frac{d}{2}\left(\frac{a'}{a}-\frac{d'}{d}\right), \tag{3.13}$$

which must be solved subject to the initial data

$$\mu(0)=0, \qquad \mu'(0)=2a(0)\neq 0 \tag{3.14}$$

in order to satisfy the initial condition for the corresponding Green function; see the asymptotic formula (3.22) below. We refer to equation (3.12) as the *characteristic equation* and its solution $\mu(t)$, subject to (3.14), as the *characteristic function.* As the special case (3.12) contains the generalized equation of hypergeometric type, whose solutions are studied in detail in [35]; see also [1], [34], [42], and [45].

Thus, the Green function (fundamental solution or propagator) is given in terms of the characteristic function

$$\psi=G(x,y,t)=\frac{1}{\sqrt{2\pi i\mu(t)}}\ e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2+\delta(t)x+\varepsilon(t)y+\kappa(t)\right)}. \tag{3.15}$$

Here

$$\alpha(t)=\frac{1}{4a(t)}\frac{\mu'(t)}{\mu(t)}-\frac{d(t)}{2a(t)}, \tag{3.16}$$

$$\beta(t)=-\frac{1}{\mu(t)}\ \exp\left(-\int_0^t(c(\tau)-2d(\tau))\ d\tau\right), \tag{3.17}$$

$$\begin{aligned}\gamma(t) &= \frac{a(t)}{\mu(t)\mu'(t)}\ \exp\left(-2\int_0^t(c(\tau)-2d(\tau))\ d\tau\right) \\ &\quad -4\int_0^t\frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2}\left(\exp\left(-2\int_0^\tau(c(\lambda)-2d(\lambda))\ d\lambda\right)\right)\ d\tau,\end{aligned} \tag{3.18}$$

$$\begin{aligned}\delta(t) &= \frac{1}{\mu(t)}\ \exp\left(-\int_0^t(c(\tau)-2d(\tau))\ d\tau\right) \\ &\quad \times\int_0^t\exp\left(\int_0^\tau(c(\lambda)-2d(\lambda))\ d\lambda\right) \\ &\quad\quad \times\left(\left(f(\tau)-\frac{d(\tau)}{a(\tau)}g(\tau)\right)\mu(\tau)+\frac{g(\tau)}{2a(\tau)}\mu'(\tau)\right)\ d\tau,\end{aligned} \tag{3.19}$$

$$
\begin{aligned}
\varepsilon(t) &= -\frac{2a(t)}{\mu'(t)}\delta(t)\ \exp\left(-\int_0^t (c(\tau)-2d(\tau))\ d\tau\right) \\
&\quad +8\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2}\exp\left(-\int_0^\tau (c(\lambda)-2d(\lambda))\ d\lambda\right)(\mu(\tau)\delta(\tau))\ d\tau \\
&\quad +2\int_0^t \frac{a(\tau)}{\mu'(\tau)}\exp\left(-\int_0^\tau (c(\lambda)-2d(\lambda))\ d\lambda\right)\left(f(\tau)-\frac{d(\tau)}{a(\tau)}g(\tau)\right)\ d\tau,
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
\kappa(t) &= \frac{a(t)\mu(t)}{\mu'(t)}\delta^2(t)-4\int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2}(\mu(\tau)\delta(\tau))^2\ d\tau \\
&\quad -2\int_0^t \frac{a(\tau)}{\mu'(\tau)}(\mu(\tau)\delta(\tau))\left(f(\tau)-\frac{d(\tau)}{a(\tau)}g(\tau)\right)\ d\tau
\end{aligned} \tag{3.21}
$$

with $\delta(0)=g(0)/(2a(0))$, $\varepsilon(0)=-\delta(0)$, and $\kappa(0)=0$. Integration by parts has been used to resolve the singularities of the initial data. Then the corresponding asymptotic formula is

$$
G(x,y,t)=\frac{e^{iS(x,y,t)}}{\sqrt{2\pi i\mu(t)}}\sim\frac{1}{\sqrt{4\pi i a(0)t}}\exp\left(i\frac{(x-y)^2}{4a(0)t}\right)\exp\left(i\frac{g(0)}{2a(0)}(x-y)\right) \tag{3.22}
$$

as $t\to 0^+$. Notice that the first term on the right hand side is a familiar free particle propagator (cf. (6.1) below).

By the superposition principle, an explicit solution of the Cauchy initial value problem

$$
i\frac{\partial\psi}{\partial t}=H(t)\psi,\qquad \psi(x,t)|_{t=0}=\varphi(x) \tag{3.23}
$$

on the infinite interval $-\infty<x<\infty$ with the general quadratic Hamiltonian as in (3.1) has the form

$$
\psi(x,t)=\int_{-\infty}^{\infty}G(x,y,t)\ \psi(y,0)\ dy. \tag{3.24}
$$

This yields the time evolution operator (2.2) explicitly as an integral operator.

4. Inverses of the Time Evolution Operator

In the previous section we have discussed how to construct the time evolution operator for the linear Schrödinger equation with the quadratic Hamiltonian (3.1). In this section we study the inverses

$$
\psi(x,t) = U(t)\psi(x,0)=\int_{-\infty}^{\infty}G(x,y,t)\ \psi(y,0)\ dy, \tag{4.1}
$$

$$
\psi(x,0) = U^{-1}(t)\psi(x,t)=\int_{-\infty}^{\infty}H(x,y,t)\ \psi(y,t)\ dy \tag{4.2}
$$

such that

$$
U(t)U^{-1}(t)=U^{-1}(t)U(t)=I=\mathrm{id}. \tag{4.3}
$$

Here we introduce

$$
G(x,y,t) = \frac{e^{iS(x,y,t)}}{\sqrt{2\pi i\mu(t)}}, \tag{4.4}
$$

$$\begin{aligned} H(x,y,t) &= G^{*}(y,x,t)\exp\left(-\int_0^t \left(c(\tau)-2d(\tau)\right)\ d\tau\right) \\ &= \frac{e^{-iS(y,x,t)}}{\sqrt{-2\pi i\mu(t)}}\exp\left(-\int_0^t \left(c(\tau)-2d(\tau)\right)\ d\tau\right) \end{aligned} \tag{4.5}$$

with $S(x,y,t)=\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2+\delta(t)x+\varepsilon(t)y+\kappa(t)$.

First we formally prove the following orthogonality relations of the kernels

$$\int_{-\infty}^{\infty} G(x,y,t)H(y,z,t)\ dy = e^{i(\alpha(t)(x+z)+\delta(t))(x-z)}\ \delta(x-z), \tag{4.6}$$

$$\int_{-\infty}^{\infty} H(x,y,t)G(y,z,t)\ dy = e^{-i(\gamma(t)(x+z)+\varepsilon(t))(x-z)}\ \delta(x-z), \tag{4.7}$$

where $\delta(x)$ is the Dirac delta function with respect to the space coordinates (do not confuse with a given function of time $\delta(t)$ throughout the paper).

Indeed, by (4.4)–(4.5) one gets

$$\begin{aligned} \int_{-\infty}^{\infty} G(x,y,t)H(y,z,t)\ dy &= \frac{1}{\mu(t)}\exp\left(-\int_0^t \left(c(\tau)-2d(\tau)\right)\ d\tau\right) \\ &\quad \times e^{i(\alpha(t)(x+z)+\delta(t))(x-z)}\ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\beta(t)(x-z)y}\ dy \\ &= e^{i(\alpha(t)(x+z)+\delta(t))(x-z)}\ \delta(x-z) \end{aligned}$$

in view of (3.17) with $-\beta(t)>0$ and the integral

$$\delta(\zeta)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\zeta\xi}\ d\xi \tag{4.8}$$

as the Dirac delta function. The formal proof of (4.7) is similar and is left to the reader.

Now we have

$$\begin{aligned} & U^{-1}(t)U(t)\psi(x,0)=U^{-1}(t)\psi(x,t) \\ &= \int_{-\infty}^{\infty} H(x,y,t)\ \ \psi(y,t)\ dy \\ &= \int_{-\infty}^{\infty} H(x,y,t)\ \left(\int_{-\infty}^{\infty} G(y,z,t)\ \ \psi(z,0)\ dz\right)\ dy \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} H(x,y,t)G(y,z,t)\ dy\right)\ \psi(z,0)\ dz \\ &= \int_{-\infty}^{\infty} e^{-i(\gamma(t)(x+z)+\varepsilon(t))(x-z)}\ \delta(x-z)\ \ \psi(z,0)\ dz \\ &= \psi(x,0), \end{aligned}$$

or $U^{-1}(t)U(t)=I$. A formal proof of the second relation $U(t)U^{-1}(t)=I$ is similar and left to the reader.

An integral form (2.6) of the nonlinear Schrödinger equation (1.2) contains also the integral operator $U(t,s) = U(t)\, U^{-1}(s)$:

$$U(t)\, U^{-1}(s)\, \psi(x,s) = \int_{-\infty}^{\infty} G(x,y,t,s)\, \psi(y,s)\ dy \tag{4.9}$$

with the kernel given by

$$G(x,y,t,s) = \int_{-\infty}^{\infty} G(x,z,t)\, H(z,y,s)\ dz. \tag{4.10}$$

Here

$$\begin{aligned}\int_{-\infty}^{\infty} G(x,z,t)\, H(z,y,s)\ dz &= \frac{1}{2\pi\sqrt{\mu(t)\,\mu(s)}} \\ &\times \exp\left(-\int_0^s \left(c(\tau) - 2d(\tau)\right)\ d\tau\right) \\ &\times e^{i\left(\alpha(t)x^2 - \alpha(s)y^2 + \delta(t)x - \delta(s)y + \kappa(t) - \kappa(s)\right)} \\ &\times \int_{-\infty}^{\infty} e^{i\left((\gamma(t)-\gamma(s))z^2 + (\beta(t)x - \beta(s)y + \varepsilon(t) - \varepsilon(s))z\right)}\ dz\end{aligned} \tag{4.11}$$

and with the help of the familiar elementary integral

$$\int_{-\infty}^{\infty} e^{i\left(az^2 + 2bz\right)}\, dz = \sqrt{\frac{\pi i}{a}}\, e^{-ib^2/a}, \tag{4.12}$$

see Refs. [3] and [36], we get

$$\begin{aligned}G(x,y,t,s) &= \frac{1}{\sqrt{4\pi i \mu(t)\,\mu(s)\,(\gamma(s) - \gamma(t))}} \\ &\times \exp\left(-\int_0^s \left(c(\tau) - 2d(\tau)\right)\ d\tau\right) \\ &\times \exp\left(i\left(\alpha(t)\,x^2 - \alpha(s)\,y^2 + \delta(t)\,x - \delta(s)\,y + \kappa(t) - \kappa(s)\right)\right) \\ &\times \exp\left(\frac{\left(\beta(t)\,x - \beta(s)\,y + \varepsilon(t) - \varepsilon(s)\right)^2}{4i\left(\gamma(t) - \gamma(s)\right)}\right).\end{aligned} \tag{4.13}$$

This can be transform into a somewhat more convenient form

$$\begin{aligned}G(x,y,t,s) = &\frac{1}{\sqrt{4\pi i \mu(t)\,\mu(s)\,(\gamma(s) - \gamma(t))}} \exp\left(-\int_0^s \left(c(\tau) - 2d(\tau)\right)\ d\tau\right) \\ \times &\exp\left(\frac{\left(\varepsilon(t) - \varepsilon(s)\right)^2 - 4\left(\gamma(t) - \gamma(s)\right)\left(\kappa(t) - \kappa(s)\right)}{4i\left(\gamma(t) - \gamma(s)\right)}\right) \\ \times &\exp\left(\frac{\left(\varepsilon(t) - \varepsilon(s)\right)\left(\beta(t)\,x - \beta(s)\,y\right) - 2\left(\gamma(t) - \gamma(s)\right)\left(\delta(t)\,x - \delta(s)\,y\right)}{2i\left(\gamma(t) - \gamma(s)\right)}\right) \\ \times &\exp\left(\frac{\left(\beta(t)\,x - \beta(s)\,y\right)^2 - 4\left(\gamma(t) - \gamma(s)\right)\left(\alpha(t)\,x^2 - \alpha(s)\,y^2\right)}{4i\left(\gamma(t) - \gamma(s)\right)}\right).\end{aligned} \tag{4.14}$$

In the limit $s \to t$ with $s < t$ one arrives at the kernel of the identity operator. We leave the details to the reader.

5. Axillary Tools: An Estimate and a Functional Equation

Consider the operator $U(t,s)=U(t)\,U^{-1}(s)$. From (4.9) and (4.14) one gets

$$\begin{aligned}
\left|U(t,s)\,\psi(x,s)\right| &= \left|\int_{-\infty}^{\infty} G(x,y,t,s)\,\psi(y,s)\ dy\right| \\
&\leq \int_{-\infty}^{\infty} \left|G(x,y,t,s)\,\psi(y,s)\right|\ dy \\
&= \frac{1}{\sqrt{4\pi\left|\mu(t)\,\mu(s)\,(\gamma(s)-\gamma(t))\right|}} \\
&\quad\times\exp\left(-\int_0^s (c(\tau)-2d(\tau))\ d\tau\right) \\
&\quad\times\int_{-\infty}^{\infty}\left|\psi(y,s)\right|\ dy
\end{aligned}$$

and as a result

$$\begin{aligned}
\left|U(t,s)\,\psi(x,s)\right| &\leq \frac{1}{\sqrt{4\pi\left|\mu(t)\,\mu(s)\,(\gamma(s)-\gamma(t))\right|}} \\
&\quad\times\exp\left(-\int_0^s (c(\tau)-2d(\tau))\ d\tau\right) \\
&\quad\times\int_{-\infty}^{\infty}\left|\psi(y,s)\right|\ dy.
\end{aligned}\tag{5.1}$$

Thus the familiar estimate

$$\begin{aligned}
\left\|U(t,s)\,\psi\right\|_{\infty} &\leq \frac{1}{\sqrt{4\pi\left|\mu(t)\,\mu(s)\,(\gamma(s)-\gamma(t))\right|}} \\
&\quad\times\exp\left(-\int_0^s (c(\tau)-2d(\tau))\ d\tau\right)\ \left\|\psi\right\|_1
\end{aligned}\tag{5.2}$$

holds in the case of the general quadratic Hamiltonian (3.1) (cf. [26]).

As we shall see in the next section, some solutions of the characteristic equation (3.12) obey the following property

$$\mu(t)\,\mu(s)\,(\gamma(s)-\gamma(t))=\chi\left(\frac{t+s}{2}\right)\mu(t-s).\tag{5.3}$$

Then

$$\mu(t)\,\mu(s)\,\frac{\gamma(s)-\gamma(t)}{t-s}=\chi\left(\frac{t+s}{2}\right)\frac{\mu(t-s)-\mu(0)}{t-s}$$

and in the limit $s\to t$ one gets

$$-\mu^2(t)\,\frac{d\gamma}{dt}=\chi(t)\,\mu'(0),$$

or by (3.14)

$$\frac{d\gamma}{dt}+2a(0)\,\frac{\chi(t)}{\mu^2(t)}=0.\tag{5.4}$$

But, in view of (3.8) and (3.17),

$$\frac{d\gamma}{dt} + \exp\left(-2\int_0^t \left(c\left(\tau\right) - 2d\left(\tau\right)\right)\ d\tau\right)\ \frac{a\left(t\right)}{\mu^2\left(t\right)} = 0. \tag{5.5}$$

Therefore, the addition property (5.3) may hold only when

$$\chi\left(t\right) = \frac{1}{2}\exp\left(-2\int_0^t \left(c\left(\tau\right) - 2d\left(\tau\right)\right)\ d\tau\right)\ \frac{a\left(t\right)}{a\left(0\right)}. \tag{5.6}$$

Some examples will be given in the next section.

6. Examples

Now let us consider several elementary solutions of the characteristic equation (3.12); more complicated cases may include special functions, like Bessel, hypergeometric or elliptic functions [1], [35], [37], and [45]. Among special cases of general expressions for the Green function (3.15)–(3.21) are the following [15]:

6.1. **A Free Particle.** When $a = 1/2$, $b = c = d = f = g = 0$, and $\mu'' = 0$, $\mu = t$, one gets

$$G\left(x, y, t\right) = \frac{1}{\sqrt{2\pi i t}}\ \exp\left(\frac{i\left(x - y\right)^2}{2t}\right) \tag{6.1}$$

as the free particle propagator [21]. In this case $\alpha = -\beta/2 = \gamma = 1/\left(2t\right)$ and an elementary identity

$$\frac{\left(x/t - y/s\right)^2}{\left(1/t - 1/s\right)} - \frac{x^2}{t} + \frac{y^2}{s} = -\frac{\left(x - y\right)^2}{t - s}$$

implies that

$$G\left(x, y, t, s\right) = G\left(x, y, t - s\right) \tag{6.2}$$

from the general formula (4.14). The time evolution operator of the linear problem is given explicitly as the following integral operator

$$U\left(t\right)\chi\left(x\right) = \frac{1}{\sqrt{2\pi i t}}\ \int_{-\infty}^{\infty} e^{i(x-y)^2/2t}\ \chi\left(y\right)\ dy \tag{6.3}$$

and the traditional nonlinear Schrödinger equation

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi = \lambda\left|\psi\right|^{2\nu}\psi, \qquad \lambda = \text{constant}, \quad 0 < \nu \leq 1 \tag{6.4}$$

has the familiar integral form

$$\psi\left(x, t\right) = U\left(t\right)\psi\left(x, 0\right) - i\lambda\int_0^t U\left(t - s\right)\left|\psi\left(x, s\right)\right|^{2\nu}\psi\left(x, s\right)\ ds. \tag{6.5}$$

See [14], [16], [26], [44] and references therein for more information.

6.2. **A Particle in a Uniform External Field.** For a particle in a constant external field, where $a = 1/2$, $b = c = d = g = 0$ and $f =$ constant, $\mu = t$, the propagator of the linear problem is given by

$$G(x, y, t) = \frac{1}{\sqrt{2\pi i t}} \exp\left(\frac{i(x-y)^2}{2t}\right) \exp\left(\frac{if(x+y)}{2}t - \frac{if^2}{24}t^3\right). \tag{6.6}$$

This case was studied in detail in [2], [6], [21], [24], [33] and [40]. We have corrected a typo in [21]; see [41] for a complete list of known errata in the Feynman and Hibbs book.

In this case once again $\alpha = -\beta/2 = \gamma = 1/(2t)$ and, in addition to the case of a free particle, $\delta = \varepsilon = (ft)/2$ and $\kappa = -f^2t^3/24$. An elementary calculation shows from the general expression (4.14) that relation (6.2) holds. Therefore, the corresponding nonlinear Schrödinger equation

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + fx\right)\psi = \lambda |\psi|^{2\nu}\psi, \qquad \lambda = \text{constant}, \quad 0 < \nu \leq 1 \tag{6.7}$$

has the integral form (6.5), where the linear propagator is given by (6.6).

In a more general case of a particle in a uniform electric field changing in time with a similar velocity-dependent term

$$\left(i\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + f(t)x - ig(t)\frac{\partial}{\partial x}\right)\psi = \lambda |\psi|^{2\nu}\psi, \tag{6.8}$$

where $f(t)$ and $g(t)$ are functions of time only, the propagator of the linear problem has the form

$$G(x, y, t) = \frac{1}{\sqrt{2\pi i t}} \exp\left(\frac{i(x-y)^2}{2t}\right) \exp\left(i\left(\delta(t)x + \varepsilon(t)y + \kappa(t)\right)\right) \tag{6.9}$$

with

$$\delta(t) = \frac{1}{t}\int_0^t \left(f(\tau)\tau + g(\tau)\right)\, d\tau, \tag{6.10}$$

$$\varepsilon(t) = -\delta(t) + \int_0^t f(\tau)\, d\tau \tag{6.11}$$

and

$$\kappa(t) = \frac{t}{2}\delta^2(t) - \int_0^t \tau\delta(t) f(\tau)\, d\tau. \tag{6.12}$$

A semigroup property [14], related to (6.2), does not hold anymore and one has to use a general expression (4.14) in order to write the integral equation (2.6). But, in view of an elementary identity

$$\mu(t)\mu(s)\left(\gamma(s) - \gamma(t)\right) = \frac{1}{2}\mu(t-s), \tag{6.13}$$

an important addition formula still holds for the amplitude of the kernel $G(x, y, t, s)$ in (4.9)–(4.10) and (4.14) of the operator $U(t, s) = U(t)U^{-1}(s)$.

6.3. **The Forced Harmonic Oscillator.** The simple harmonic oscillator with $a = b = 1/2$, $c = d = f = g = 0$ and $\mu'' + \mu = 0$, $\mu = \sin t$ has the familiar propagator of the form

$$G(x,y,t) = \frac{1}{\sqrt{2\pi i \sin t}} \exp\left(\frac{i}{2\sin t}\left(\left(x^2+y^2\right)\cos t - 2xy\right)\right), \tag{6.14}$$

which is studied in detail at [4], [22], [25], [30], [32], [43].

Once again relation (6.2) holds and the corresponding nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\right)\psi = \lambda\left|\psi\right|^{2\nu}\psi, \qquad \lambda = \text{constant}, \quad 0 < \nu \leq 1 \tag{6.15}$$

has the integral form (6.5), where the propagator is given by (6.14). See [14], [8], [9], [10], [11], [12], [13] and references therein for investigation of solutions of this integral equation by a fixed point argument.

A linear problem extension to the case of the forced harmonic oscillator including an extra velocity-dependent term and a time-dependent frequency is discussed in [17], [18], [21] and [29]. The nonlinear Schrödinger equation of interest is

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\right)\psi + f(t)\,x\psi - ig(t)\frac{\partial \psi}{\partial x} = \lambda\left|\psi\right|^{2\nu}\psi \tag{6.16}$$

and the corresponding propagator has the form

$$\begin{aligned} G(x,y,t) &= \frac{1}{\sqrt{2\pi i \sin t}} \exp\left(\frac{i}{2\sin t}\left(\left(x^2+y^2\right)\cos t - 2xy\right)\right) \\ &\quad \times \exp\left(i\left(\delta(t)\,x + \varepsilon(t)\,y + \kappa(t)\right)\right) \end{aligned} \tag{6.17}$$

with

$$\delta(t) = \frac{1}{\sin t}\int_0^t \left(f(\tau)\sin\tau + g(\tau)\cos\tau\right)\,d\tau, \tag{6.18}$$

$$\varepsilon(t) = -\frac{\delta(t)}{\cos t} + \int_0^t \frac{\sin\tau\;\delta(\tau)}{\cos^2\tau}\,d\tau + \int_0^t \frac{f(\tau)}{\cos\tau}\,d\tau \tag{6.19}$$

and

$$\begin{aligned} \kappa(t) &= \frac{1}{2}\tan t\;\delta^2(t) - \frac{1}{2}\int_0^t \tan^2\tau\;\delta^2(\tau)\,d\tau \\ &\quad - \int_0^t \tan\tau\;\delta(\tau)\,f(\tau)\,d\tau. \end{aligned} \tag{6.20}$$

The addition property (5.3) holds in this case. We leave the detail to the reader.

6.4. **A Modified Oscillator.** Furthermore, an exact solution of the n-dimensional time-dependent Schrödinger equation for certain modified oscillator is found in [31]. In the one-dimensional case we get functions $a = \cos^2 t$, $b = \sin^2 t$, $c = 2d = \sin 2t$ and our characteristic equation (3.12) takes the form

$$\mu'' + 2\tan t\;\mu' - 2\mu = 0, \tag{6.21}$$

whose elementary solution $\mu = \cos t \sinh t + \sin t \cosh t$ satisfies the initial conditions (3.14). Further, the corresponding propagator is given by

$$G(x,y,t) = \frac{1}{\sqrt{2\pi i \left(\cos t \sinh t + \sin t \cosh t\right)}} \times \exp\left(\frac{\left(x^2-y^2\right)\sin t \sinh t + 2xy - \left(x^2+y^2\right)\cos t \cosh t}{2i\left(\cos t \sinh t + \sin t \cosh t\right)}\right), \tag{6.22}$$

which was found in [31] as the special case $n=1$ of a general n-dimensional expansion of the Green function in hyperspherical harmonics. We have showed that (6.22) is a generalization of the propagator for the simple harmonic oscillator; see Ref. [31] for more details.

The corresponding nonlinear Schrödinger equation [15]

$$i\frac{\partial \psi}{\partial t} + \cos^2 t\ \frac{\partial^2 \psi}{\partial x^2} - \sin^2 t\ x^2\psi + i \sin t \cos t \left(2x\frac{\partial \psi}{\partial x} + \psi\right) = h(t)\left|\psi\right|^{2\nu}\psi \tag{6.23}$$

can be rewritten in an integral form. We leave the detail to the reader.

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